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Stability of conserving approximations and second-order phase transitions in interacting Fermi systems

G. MORANDI†

Department of Physics, Imperial College

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Abstract. The equivalence between the thermodynamic stability condition that the free energy be a local minimum and the requirement that the correlation functions show no zero-frequency poles is proved for a general class of conserving approximations to many-fermion problems. Previous stability treatments are rederived as particular cases of the general theorem, and some applications are developed. The consequences of the theorem are briefly discussed.

1. Introduction

The problem of the thermodynamic stability of an interacting many-fermion system is usually attacked from two different points of view. The first one is a purely statistical-mechanical approach, in which one starts with a given Hamiltonian or a given effective Hamiltonian and builds up the partition function and the free energy, requiring then the latter to be a (at least relative) minimum. This approach has the advantage of being mathematically very straightforward, once a good mathematical characterization of the Hamiltonian has been given; in particular, it allows one to establish a number of very elegant and useful properties, such as minimum principles and absolute lower bounds for any approximation to the free energy (see, e.g., Huber 1967). Its disadvantage is that it is almost never possible to define a clear-cut effective Hamiltonian corresponding to a given approximation, except for the very simple ones. On the other hand, the Green function approach allows one to develop a whole class of approximations which are much more useful in actual descriptions of a Fermi system; the stability problem comes out in this formalism as the requirement that certain correlation functions (and the susceptibilities that can be deduced from them) develop no complex poles corresponding to collective modes of the system growing exponentially in time. The onset of instabilities is usually said to be at the point where such functions develop a zero-frequency pole. Examples of this kind are the well-known Stoner criterion for itinerant ferromagnetism (Stoner 1938) and the t -matrix approach to the Cooper instability (Schrieffer 1964). From a conservative point of view the request of no complex or zero-frequency poles appears rather as a requirement that the perturbation expansion for the relevant physical response functions be well behaved and convergent, so that we can end up with results displaying the correct analytical properties. There is no general clear-cut connection of this with the requirement that the free energy be a minimum. Such a connection, as far as we know, has been established only in two cases, namely in the Hartree-Fock approximation (Thouless 1960, Mermin 1963) and for the Landau-Fermi liquid theory (Pomeranchuk 1958). Both suffer from limitations, the former being restricted to the simplest approximation of the many-body approach, the latter being limited to the region of temperatures and momenta where the Landau theory is valid. It is the purpose of this paper to show how this connection can be established, in the form of a strict equivalence, holding for a general class of Green function approximations (the so-called 'conserving' approximations, which include the exact theory). We shall derive the stability condition by analysing the second variation of the free energy, considered as a functional of the Green function, around the value of the latter representing the normal-state solution (e.g. for an interacting electron gas this will be the (fully interacting) uniform paramagnetic state) when the Green function itself, or the self-energy, undergoes a small variation.

† On leave from Istituto Nazionale di Fisica Nucleare, Bologna, Italy.

As a typical such variation we could consider, for example, the off-diagonal components of the Nambu matrix Green function in the theory of superconductivity (Nambu 1960, Schrieffer 1964) or the anomalous Green functions (off-diagonal in both momentum and spin) which must be introduced to treat the spin density wave itinerant antiferromagnet (Celli and Morandi 1967).

The theory is not limited to any particular example; the only assumption is that such 'anomalous' Green functions behave continuously and are zero in the normal phase. We then essentially restrict ourselves to second-order phase transitions, which are characterized by the continuous growing up of an order parameter (intimately connected with the anomalous Green functions) from zero in the normal phase to a finite value. Two kinds of techniques can be used: the perturbation-theoretic one, making use of Feynman's diagrams, and the (in principle, non-perturbative) functional derivative technique of Baym and Kadanoff (Baym and Kadanoff 1962). The latter can be given, step by step, a diagrammatic interpretation (which is perhaps more intuitively clear). We choose to use this second kind of technique; the diagrammatic approach is used either to illustrate the basic equations, or to prove some simple property, when the proof turns out to be shorter and simpler.

In § 2 we summarize the basic theory to be used throughout the paper; the stability criterion is established in § 3 and the connection with the zero-frequency behaviour of the correlation functions is established in § 4. Section 5 is devoted to rederiving the Mermin-Thouless theorem as a particular case and to some simple applications. In § 6 we state the conclusions. The appendix contains the proof of a relation which is of use in the text.

2. Conserving approximations in the many-fermion problem

In this section we briefly summarize the basic formal apparatus to be used throughout the paper. All the results we shall establish (or simply state) can be found in the literature (mainly in the papers of Baym and Kadanoff (1961, 1962) and Baym (1962)) and are collected here only to establish the notation and some minor conventions.

Let \hat{H}_0 be the full Hamiltonian (kinetic plus potential energy) of the system and μ the chemical potential; we find it convenient to employ, as a time-development operator,

$$\hat{H} = \hat{H}_0 - \mu \hat{N} \quad (2.1)$$

(\hat{N} being the particle number operator) instead of H_0 . The one-particle Green function is defined as

$$G(1,1') = \frac{1}{i} \langle T\{\psi(1)\psi^+(1')\} \rangle \quad (2.2)$$

where ψ, ψ^+ are Heisenberg wave-field operators, 1, 1' stand for coordinate, time and spin indices, the times are assumed to run on the interval $(0, -i\beta)$ (Baym and Kadanoff 1962), $\langle \dots \rangle$ means a statistical average (in the grand canonical ensemble, i.e. the canonical ensemble of H) and T is the usual Dyson time-ordering operator (which orders with respect to the real variables it_1, it_1'). If we assume that an external non-local potential $U(1,2)$ is acting on the system, G can be written, in the 'interaction' representation in which H is the zero-order Hamiltonian, as

$$G(1,1') = \frac{1}{i} \frac{\langle T[S\psi(1)\psi^+(1')] \rangle}{\langle T[S] \rangle} \quad (2.3)$$

where

$$S = T \exp \left\{ -i \int d(1) d(2) \psi^+(1) U(1,2) \psi(2) \right\} \quad (2.4)$$

and $\int d(i)$ stands for $\sum_{\text{spin}} \int d^3r_i \int_0^{-i\beta} dt_i$. In both cases G satisfies the boundary condition

$$G(1,1')|_{t_1=0} = -G(1,1')|_{t_1=-i\beta}. \quad (2.5)$$

The Dyson equation for G in the presence of U is

$$G^{-1}(1,1') = G_0^{-1}(1,1') - U(1,1') - \Sigma(1,1')$$

$$G_0^{-1}(1,1') = \left(i \frac{\partial}{\partial t_1} - \hat{h}(1) + \mu \right) \delta(1-1') \tag{2.6}$$

where $h(1)$ is the one-particle part (kinetic energy plus, eventually, a one-body potential) of H_0 and Σ the proper self-energy. A knowledge of G as a functional of U allows one to determine the two-particle Green function $G_2(1, 1'; 2, 2')$ as

$$G_2(1,2;1'2') = \left(\frac{1}{i} \right)^2 \langle T \{ \psi(1)\psi(2)\psi^+(2')\psi^+(1') \} \rangle = - \left. \frac{\delta G(1,1')}{\delta U(2',2)} \right|_{U=0} + G(1,1')G(2,2'). \tag{2.7}$$

G itself can be determined from a knowledge of the functional (Baym 1962)

$$Z = \text{Sp} \{ \exp(-\beta H) T[S] \} \tag{2.8}$$

which is the appropriate generalization to our case of the well-known expression for the partition function. If we set

$$W = -\ln Z \tag{2.8'}$$

G is given by

$$G(1,1') = \frac{\delta W}{\delta U(1',1)}. \tag{2.9}$$

W can, in turn, be expressed as a functional of G and U (Baym 1962, Luttinger and Ward 1960, see also Nozières 1964):

$$W = \Phi - \text{Tr} \{ (G_0^{-1} - U)G - 1 \} + \text{Tr} \ln(-G) \tag{2.10}$$

where the functional Φ has the property that

$$\frac{\delta \Phi}{\delta G(1',1)} = \Sigma(1,1'). \tag{2.11}$$

Products like UG are to be understood as

$$(UG)(1,2) = \int d(3)U(1,3)G(3,2)$$

and

$$\text{Tr} A = \int d(1)A(1,1^+) \tag{2.12}$$

($1^+ \equiv (\mathbf{r}_1, t_1 + 0)$). W has the property of being stationary with respect to G ; in fact by setting $\delta W / \delta G = 0$ we simply reproduce Dyson's equation. Equation (2.9) is easily derived from (2.10); owing to the stationarity property, we need only to vary W with respect to its explicit dependence on U appearing in the term $\text{Tr} UG$, which immediately leads to (2.9). An explicit diagrammatic expression for Φ can be given if we analyse Σ into skeleton diagrams (i.e. diagrams from which all the diagonal self-energy insertions have been removed and full propagators replace the bare ones). Let $\Sigma^{(n)}$ (itself a functional of G and U) be the sum of all such diagrams containing n vertices; then (Nozières 1964)

$$\Phi = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \{ \Sigma^{(n)} G \}. \tag{2.13}$$

These are the more relevant features of the exact theory. Needless to say it satisfies all the physically important conservation laws (particle number, energy, momentum, angular

momentum). The elegance of this formulation is in the fact that (Baym 1962) all the conservation laws follow from them being satisfied at each vertex (microscopic conservation) and from the single property of the self-energy of being the variational derivative of a functional of the exact G .

To be conserving (i.e. to satisfy the conservation laws) is a requirement to be imposed on every physically meaningful approximation. Equations (2.1) to (2.12) define a whole class of conserving approximations, each one being uniquely defined by giving an explicit expression for the functional Φ , which in turn uniquely defines W and Σ . Through (2.6) and (2.7) the one- and two-particle fully conserving Green functions can be calculated; at the end of the calculation U can be set equal to zero and then W gives the approximation to the grand potential consistent with the given Green functions. It is this class of conserving (or ' Φ -derivable' (Baym 1962)) approximations we are interested in. Examples of conserving approximations, including the well-known ones (Hartree and Hartree-Fock, random phase approximation, etc.) are given in the literature quoted in this section and will not be reproduced here.

A property of conserving approximations which is of great use and which we shall need later is that they preserve all the thermodynamic relations that arise from varying a parameter in the grand potential (Baym 1962). The stationarity property of W , which allows us to vary only with respect to its explicit dependence on such a parameter, is of great use in performing the calculation. One such relation which is easily deduced from (2.10) is the thermodynamic relation

$$\frac{\partial}{\partial \mu} \ln Z = \beta \langle \hat{N} \rangle \quad (2.14)$$

($W = -\ln Z$) which gives

$$\langle \hat{N} \rangle = \frac{1}{\beta} \text{Tr} G = -i \int d^3r G(rt; rt^+) \quad (2.15)$$

consistent with the usual definition of $\langle \hat{N} \rangle$.

A last formula which will be useful later connects the variation of the self-energy with the vertex part. If we define (Baym and Kadanoff 1961)

$$\Gamma(1,1';2,2') = \left. \frac{\delta \Sigma(1,2)}{\delta G(2',1')} \right|_{U=0} \quad (2.16)$$

it is easy to convince oneself that Γ is just the irreducible particle-hole vertex part of the standard many-body perturbation theory.

3. The stability criterion

In all that follows we only assume that a well-defined functional Φ exists and satisfies (2.11); our conclusions hold then for any conserving approximation.

Let us suppose the programme sketched in the previous section has been pushed to the end and U set equal to zero. We are then describing thermal equilibrium, and $\Omega = \beta^{-1}W$, from (2.10), is a definite approximation to the grand potential Ω . By varying around one of its stationary points (the one we have chosen to represent the Green function) given by Dyson's equation, we obtain (the first variation being zero)†

$$\delta^2 \Omega = \frac{1}{2\beta} \text{Tr} \left\{ \delta G \text{Tr} \frac{\delta^2 \Phi}{\delta G \delta G'} \delta G' - (G^{-1} \delta G)^2 \right\} \quad (3.1)$$

the first term being a shorthand notation for

$$\int \delta G(1,2) \frac{\delta^2 \Phi}{\delta G(1,2) \delta G(1',2')} \delta G(1'2') d(1) d(2) d(1') d(2'). \quad (3.2)$$

† After deriving equation (3.1) we found that it already existed in the literature (Baym 1962).

It is better to express δG in terms of a self-energy variation as

$$\delta G = -G \delta G^{-1} G \tag{3.3}$$

and

$$\delta G^{-1} = \delta(G_0^{-1} - \Sigma) = -\delta \Sigma. \tag{3.4}$$

If we denote the self-energy variation $\delta \Sigma$ by Δ , we have

$$\delta^2 \Omega = \frac{1}{2\beta} \text{Tr} \left\{ G \Delta G \text{Tr}' \frac{\delta^2 \phi}{\delta G \delta G'} G' \Delta' G' - (\Delta G)^2 \right\}. \tag{3.5}$$

Let us choose an orthonormal basis $\{\varphi_i(\mathbf{r})\}$ of single-particle states. In this basis the one-particle Green function can be defined as

$$G_{ij}(t_1, t_1') = \int d^3 r_1 d^3 r_1' \varphi_i^*(\mathbf{r}_1) G(1, 1') \varphi_j(\mathbf{r}_1') \tag{3.6}$$

and a similar representation can be given to the self-energy.

For a system at equilibrium G_{ij} depends only on $\tau = t_1 - t_1'$; its Fourier expansion is (Baym and Kadanoff 1962)

$$G_{ij}(\tau) = \frac{i}{\beta} \sum_{\nu} \exp(-i z_{\nu} \tau) G_{ij}(z_{\nu})$$

$$G_{ij}(z_{\nu}) = \int_0^{-i\beta} d\tau \exp(i z_{\nu} \tau) G_{ij}(\tau) \tag{3.7}$$

and z_{ν} is restricted to the values $z_{\nu} = (2\nu + 1)\pi i/\beta$, $\nu = 0, \pm 1, \pm 2, \dots$, owing to the boundary condition (2.5). Moreover, $G_{ij}(z_{\nu})$ has the spectral representation

$$G_{ij}(z_{\nu}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A_{ij}(\omega)}{z_{\nu} - \omega}. \tag{3.8}$$

If we start from (3.8), G_{ij} can be analytically continued to all complex z (Baym and Kadanoff 1962).

The spectral density is given by

$$A_{ij}(\omega) = G_{ij}^{>}(\omega) + G_{ij}^{<}(\omega) \tag{3.9}$$

$G_{ij}^{\lessgtr}(\omega)$ being the Fourier transforms of the correlation functions $\langle \psi_i(t) \psi_j^+ \rangle$ and $\langle \psi_j^+ \psi_i(t) \rangle$ respectively (with real t). The boundary condition (2.5) is equivalent to

$$G_{ij}^{>}(\omega) = e^{\beta\omega} G_{ij}^{<}(\omega). \tag{3.10}$$

From the representation (3.6) applied to $G_{ij}^{\lessgtr}(t_1, t_1')$ it is clear that for real times

$$G_{ij}^{>}(t_1 - t_1') = G_{ji}^{>*}(t_1' - t_1) \tag{3.11}$$

whence

$$G_{ij}^{>}(\omega) = G_{ji}^{>*}(\omega) \tag{3.12}$$

and hence A_{ij} will obey the hermiticity condition

$$A_{ij}(\omega) = A_{ji}^*(\omega). \tag{3.13}$$

The same must be true for Σ , the more general form of which is

$$\Sigma_{ij}(z_{\nu}) = \Sigma_{ij}^{\text{HF}} + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\gamma_{ij}(\omega)}{z_{\nu} - \omega}. \tag{3.14}$$

The static part Σ^{ij} usually arises as a result of a Hartree-Fock-like calculation (whence

the suffix HF). The above hermiticity condition will hold for γ and we must also have $\Sigma_{ij}^{HF} = (\Sigma_{ji}^{HF})^*$. To avoid specifying separately the two conditions, we state them in the equivalent form

$$\Sigma_{ij}(z_\nu) = \Sigma_{ji}^*(-z_\nu). \tag{3.15}$$

The self-energy variation Δ can be as general as possible as far as the dependence on the space and spin coordinates is concerned. We again require time-translational invariance, so that

$$\Delta(1, 2) = \Delta(\mathbf{r}_1, \mathbf{r}_2; t_1 - t_2) \tag{3.16}$$

and $\Delta_{ij}(z_\nu)$ will obey the same hermiticity requirement (3.15) as Σ and G .

The irreducible vertex part $\Gamma(11', 22')$ can be transformed in a similar way; let us define

$$\begin{aligned} \Gamma(11', 22') = & \left(\frac{i}{\beta}\right)^2 \sum_{\nu_1 \dots \nu_2'} \exp\{i(z_{\nu_2} t_2 + z_{\nu_2'} t_2' - z_{\nu_1} t_1 - z_{\nu_1'} t_1')\} \\ & \times \sum_{ijkl} \varphi_i(\mathbf{r}_1) \varphi_k(\mathbf{r}_1') \varphi_j^*(\mathbf{r}_2) \varphi_l^*(\mathbf{r}_2') \langle ik | \Gamma(\nu_1 \dots \nu_2') | jl \rangle. \end{aligned}$$

At equilibrium Γ will depend only on three independent times; we define then

$$\langle ik | \Gamma(\nu_1 \dots \nu_2') | jl \rangle = \frac{1}{\beta} \delta_{\nu_1 + \nu_1', \nu_2 + \nu_2'} \langle ik | \Gamma(\nu, \nu'; \mu) | jl \rangle \tag{3.17}$$

where

$$z_\nu = \frac{1}{2}(z_{\nu_1} + z_{\nu_2}), \quad z_{\nu'} = \frac{1}{2}(z_{\nu_1'} + z_{\nu_2'}); \quad \Omega_\mu = z_{\nu_2} - z_{\nu_1} = z_{\nu_1'} - z_{\nu_2'}.$$

Ω_μ is then a 'Bose-like' (integral multiple of $2\pi i/\beta$) frequency. Expressed in perturbation theoretic language, $\langle ik | \Gamma(\nu, \nu'; \mu) | jl \rangle$ is the sum of all the irreducible (in the particle-hole channel) connected diagrams having the structure shown in figure 1.

In this representation $\delta^2\Omega$ can be written as

$$\delta^2\Omega = \frac{1}{2\beta^2} \sum_{pqrs} \sum_{\nu\nu'} \Delta_{pq}(-z_\nu) \langle qp | \hat{X}(z_\nu, z_{\nu'}) | sr \rangle \Delta_{sr}(z_{\nu'}) \tag{3.18}$$

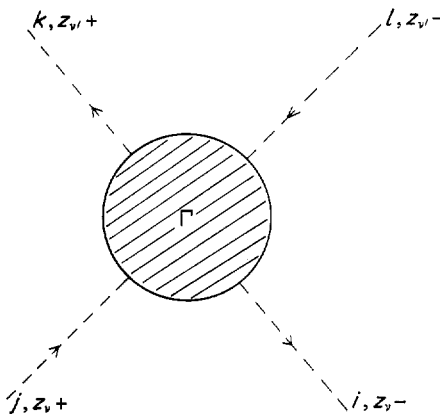


Figure 1. Graphical representation of (3.17). Here the four external lines are indicated for clarity.

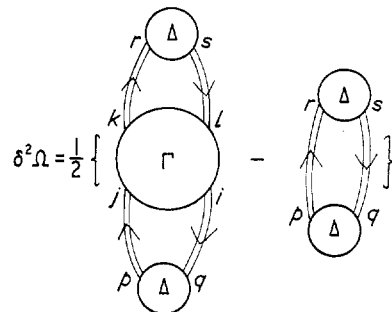


Figure 2. Graphical representation of (3.18). Double full lines represent full G 's.

where the matrix \hat{X} is defined as

$$\langle qp|\hat{X}(z_v, z_{v'})|sr\rangle = \sum_{ijkl} G_{jp}(-z_v)G_{qi}(-z_{v'})\langle ik|\Gamma(-v, v'; 0)|jl\rangle G_{is}(z_v)G_{rk}(z_{v'}) + \beta\delta_{v, -v'}G_{qs}(-z_v)G_{rp}(-z_{v'}). \tag{3.19}$$

A graphical representation of (3.18) is given in figure 2.

Let us set $z_v = i\mu, z_{v'} = i\mu'$ and indicate the frequency dependence simply by the indices μ, μ' . By using the hermiticity property (3.15), (3.18) can be rewritten as

$$\delta^2\Omega = \frac{1}{2\beta^2} \sum_{\mu\mu'} \sum_{pqrs} \Delta_{qp}^*(\mu)\langle qp|\hat{X}_{\mu\mu'}|sr\rangle\Delta_{sr}(\mu'). \tag{3.20}$$

We shall prove that \hat{X} has the following properties:

$$\langle qp|\hat{X}_{\mu\mu'}|sr\rangle = \langle sr|\hat{X}_{\mu'\mu}|qp\rangle^* \tag{hermiticity} \tag{3.21a}$$

$$\langle qp|\hat{X}_{\mu\mu'}|sr\rangle = \langle pq|\hat{X}_{-\mu, -\mu'}|rs\rangle^* \tag{inversion symmetry}. \tag{3.21b}$$

If we consider \hat{X} as a matrix in the two groups of indices $(qp\mu)$ and $(sr\mu')$, (3.21a) states that \hat{X} is Hermitian in the canonical sense of matrix theory. A diagrammatic proof of (3.21a) and (3.21b) is given in the appendix. As anticipated we shall exploit there the connection between the functional-derivative and the diagrammatic approaches. The proof could also be given by relying only on the former (i.e. exploiting the property of Γ of being a second functional derivative); the diagrammatic proof is, however, considerably simpler. Equation (3.21a) proves that, \hat{X} being Hermitian, $\delta^2\Omega$ is (as it should be) manifestly real for an arbitrary Δ satisfying the hermiticity constraint. To ensure that the solution of the thermal equilibrium problem represents a true stable equilibrium state (or at least a differentially stable one) we require that

$$\delta^2\Omega > 0. \tag{3.22}$$

Equation (3.21b) allows us to relax the Hermitian constraint imposed on Δ , for if we suppose \hat{X} has an eigenvector of the form (3.14) (but without any Hermitian constraint), $\alpha_{pq}(\mu)$ with a non-positive eigenvalue λ (λ cannot, in any case, be complex owing to (3.21a)), then we shall have

$$\frac{1}{\beta} \sum_{sr\mu'} \langle qp|\hat{X}_{\mu\mu'}|sr\rangle\alpha_{sr}(\mu') = \lambda\alpha_{qp}(\mu). \tag{3.23}$$

Using (3.21b) we also have

$$\frac{1}{\beta} \sum_{sr\mu'} \langle qp|\hat{X}_{\mu\mu'}|sr\rangle\alpha_{rs}^*(-\mu') = \lambda\alpha_{pq}^*(-\mu) \tag{3.23'}$$

but then every linear combination of $\alpha_{qp}(\mu)$ and $\alpha_{pq}^*(-\mu)$ is a solution of (3.23). By taking now

$$\Delta_{qp}(\mu) = \alpha_{qp}(\mu) + \alpha_{pq}^*(-\mu)$$

or

$$\Delta_{qp}(\mu) = i\{\alpha_{qp}(\mu) - \alpha_{pq}^*(-\mu)\} \tag{3.24}$$

which cannot be simultaneously zero, we can satisfy (3.15), but (3.22) would be violated.

Let us also note that, from the form of (3.20), we can define a bilinear form

$$\langle\alpha, \beta\rangle = \frac{1}{\beta} \sum_{pq\mu} \alpha_{pq}^*(\mu)\beta_{pq}(\mu) \tag{3.25}$$

between any two 'vectors' α, β of the given structure, and this has all the properties of a scalar product. The linear space spanned by the $\alpha_{pq}(\mu)$'s has then the correct metric

properties to enable us (taking for granted that expressions like (3.20) and (3.25) are convergent) to treat it as a Hilbert space and to consider \hat{X} as a Hermitian operator on it.

We conclude then that for a given conserving approximation to lead to a (differentially) stable thermal equilibrium solution, the Hermitian matrix $\langle qp | \hat{X}_{\mu\nu} | sr \rangle$ must be positive definite. This is the stability theorem we shall employ in the next sections.

To work with discrete imaginary frequencies has proved to be very useful in establishing the formal properties of $\delta^2\Omega$ and hence the stability criterion. At this stage and for practical purposes, we are obviously free to continue analytically, using the standard procedures, both Δ and Γ into the complex plane of frequencies and to replace the sums with integrals in which the temperature dependence is explicitly exhibited in the integrands. This has obviously to be done if one wants to evaluate $\delta^2\Omega$ for a given approximation.

4. Collective modes versus stability

We define the two-particle correlation function $L(1,2; 1'2')$ as

$$L(1, 2; 1', 2') = - \frac{\delta G(1, 1')}{\delta U(2', 2)} \tag{4.1}$$

Using the relationship between G, G^{-1} and the chain rule of functional differentiation

$$\frac{\delta}{\delta U(2', 2)} = \int d(3) d(4) \frac{\delta G(3, 4)}{\delta U(2', 2)} \frac{\delta}{\delta G(3, 4)} \tag{4.2}$$

we can generate the Bethe-Salpeter equation for L :

$$L(22', 11') = -G(2', 1)G(2, 1') + \int d(\bar{1}) \dots d(\bar{2}) G(\bar{1}, 1)G(2, \bar{2})\Gamma(\bar{2}\bar{2}', \bar{1}\bar{1}')L(\bar{1}'2', \bar{2}'1'). \tag{4.3}$$

In the basis we are using, L has a representation similar to Γ . In that basis (4.3) becomes, after Fourier transforming,

$$\begin{aligned} \langle ik | L(\nu, \nu', m) | jl \rangle &= -\beta \delta_{\nu, \nu'} G_{kj}(\mathbf{z}_{\nu+}) G_{il}(\mathbf{z}_{\nu-}) \\ &+ \frac{1}{\beta} \sum_{\nu''} \sum_{pqrs} G_{sj}(\mathbf{z}_{\nu+}) G_{iq}(\mathbf{z}_{\nu-}) \langle qp | \Gamma(\nu, \nu''; m) | sr \rangle \langle rk | L(\nu'', \nu'; m) | pl \rangle \\ &\left(\mathbf{z}_{\nu\pm} = \mathbf{z}_{\nu} \pm \frac{\Omega_m}{2}, \text{ etc.} \right). \end{aligned} \tag{4.4}$$

(4.4) is graphically represented in figure 3.

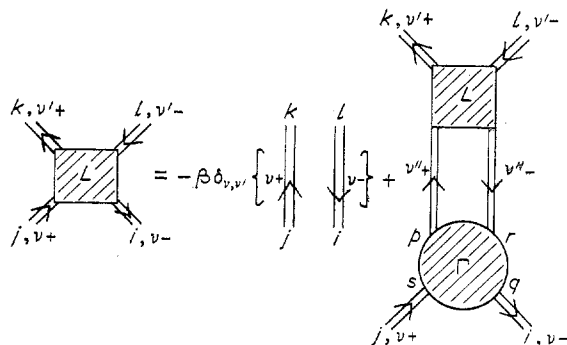


Figure 3. Bethe-Salpeter equation for L . L is represented by a shaded square with four external lines.

In the particle-hole channel Ω is the (complex) total energy of the particle-hole pair (which is conserved during the interaction). L can then be used to build up the response function of the system to an external probe varying in time with frequency Ω_m ; the physical response function is obtained by analytically continuing to all complex Ω and taking the limit as Ω tends to the real axis from above. The poles of L in the variable Ω give then the frequencies of the collective modes of the system.

We are interested here in the static limit ($\Omega_m = 0$) of the response function. We again find it more useful not to perform the analytical continuation with respect to $z_\nu, z_{\nu'}$. If we define then, in the static limit,

$$\langle ik|L(\nu, \nu', 0)|jl\rangle = \langle ik|L_{\mu\mu'}|jl\rangle \tag{4.5}$$

we have for $L_{\mu\mu'}$ the equation

$$\begin{aligned} \langle ik|L_{\mu\mu'}|jl\rangle = & -\beta\delta_{\mu\mu'}G_{kj}(\mu)G_{il}(\mu) \\ & + \frac{1}{\beta} \sum_{\mu''} \sum_{pqrs} G_{sj}(\mu)G_{iq}(\mu) \langle qp|\Gamma_{\mu\mu''}|sr\rangle \langle rk|L_{\mu''\mu'}|pl\rangle. \end{aligned} \tag{4.6}$$

If we invert the relation (3.19) between \hat{X} and Γ we obtain

$$\begin{aligned} \langle ik|\Gamma_{\mu\mu'}|jl\rangle = & \sum_{pqrs} G_{pj}^{-1}(\mu)G_{iq}^{-1}(\mu) \langle qp|\hat{X}_{-\mu, \mu'}|sr\rangle G_{sl}^{-1}(\mu')G_{kr}^{-1}(\mu') \\ & + \beta\delta_{\mu\mu'}G_{il}^{-1}(\mu)G_{kj}^{-1}(\mu). \end{aligned} \tag{4.7}$$

If we define further

$$\langle sr|\hat{\Lambda}_{\mu\mu'}|vt\rangle = \sum_{ijkl} G_{sl}^{-1}(\mu)G_{jr}^{-1}(\mu) \langle ik|L_{\mu-\mu'}|jl\rangle G_{tk}^{-1}(-\mu')G_{iv}^{-1}(-\mu') \tag{4.8}$$

(4.6) turns out to be equivalent to

$$\frac{1}{\beta} \sum_{\mu} \sum_{sr} \langle qp|\hat{X}_{\mu\bar{\mu}}|sr\rangle \langle sr|\hat{\Lambda}_{\mu\mu'}|vt\rangle = \beta\delta_{\mu\mu'}\delta_{tp}\delta_{qv} \tag{4.9}$$

i.e. to the statement that $\hat{\Lambda}$ is the inverse matrix of \hat{X} , in the sense the word ‘matrix’ has been used in the previous section.

Let us suppose now that L has a pole at $\Omega = 0$. The residue in the pole is a solution of the homogeneous equation associated with (4.6); it can be represented (Nozières 1964, p. 249) as

$$\text{Res } \langle ik|L_{\mu\mu'}|jl\rangle = U_{ij}(\mu)U_{kl}(\mu'). \tag{4.10}$$

At the same point $\hat{\Lambda}$ will also have a pole, and

$$\text{Res } \langle pq|\hat{\Lambda}_{\mu\mu'}|hk\rangle = \alpha_{pq}(\mu)\alpha_{hk}(-\mu') \tag{4.11}$$

and α will be a solution of

$$\sum_{\bar{\mu}} \sum_{pq} \langle ij|\hat{X}_{\mu\bar{\mu}}|pq\rangle \alpha_{pq}(\bar{\mu}) = 0. \tag{4.12}$$

If $\alpha \neq 0$, the stability condition is manifestly violated by (4.12). We then conclude that for a given conserving approximation to lead to a stable thermodynamic equilibrium state, no correlation function must have poles at zero frequency (with a non-trivial residue).

If instead we are in a situation in which only a weaker stability condition holds, i.e. $\delta^2\Omega \geq 0^\dagger$, the Δ which makes $\delta^2\Omega$ vanish must be a solution of (4.12), as for this the Hermitian form (3.20) assumes its minimum value.

Going back to the original definitions, we find that (4.12) is identical with

$$U_{ij}(\mu) = \frac{1}{\beta} \sum_{\bar{\mu}} \sum_{pqrs} G_{jq}(\mu)G_{pi}(\mu) \langle pr|\Gamma_{\mu\bar{\mu}}|qs\rangle U_{sr}(\bar{\mu}) \tag{4.13}$$

\dagger This can be thought of as the situation corresponding to the onset of a given instability.

i.e. the homogeneous Bethe–Salpeter equation for L at $\Omega_\mu = 0$. L in turn can be expressed in terms of the full vertex $\tilde{\Gamma}$ which is the solution of a Bethe–Salpeter equation similar to (4.3):

$$\tilde{\Gamma}(11', 22') = \Gamma(11', 22') + \int d(\bar{1}) \dots d(\bar{2}') \Gamma(1\bar{1}, 2\bar{2}) G(\bar{2}, \bar{1}') G(\bar{2}', \bar{1}) \tilde{\Gamma}(\bar{1}', 1', \bar{2}', 2'). \quad (4.14)$$

In terms of $\tilde{\Gamma}$, L is given by

$$L(11', 22') = -G(1'2)G(12') + \int d(\bar{1}) \dots d(\bar{2}') G(1\bar{1}) G(\bar{2}2) \tilde{\Gamma}(\bar{1}\bar{1}', \bar{2}\bar{2}') G(\bar{2}'2') G(1'\bar{1}') \quad (4.15)$$

(it can be checked that (4.15) is the solution of (4.3), with $\tilde{\Gamma}$ given by (4.14)). Equation (4.13) is then also equivalent to the homogeneous Bethe–Salpeter equation for $\tilde{\Gamma}$. The latter is the usual starting point for the analysis of the onset of instabilities in the normal phase. We have then shown how it comes out from, and is equivalent to, the failure of the normal-state solution to give a relative minimum of the free energy.

A couple of brief comments should be added at this point. The above statements can be somewhat disturbing at first sight, in so far as everybody can produce plenty of examples of collective modes of many-fermion systems whose frequency continuously vanishes when some parameter (in general the momentum of the mode) tends to zero without any consequences at all for the stability. This is the case, for example, for the zero and first-sound modes in a fermion system with short-range repulsive interaction. Let us fix our attention on this case (i.e. $\Omega = \Omega(\mathbf{q})$ and $\lim_{\mathbf{q} \rightarrow 0} \Omega(\mathbf{q}) = 0$); then it can be easily verified (cf. Nozières and Pines 1966) that the residue in the pole of the appropriate response function *vanishes* when $\mathbf{q} \rightarrow 0$ (i.e. this is a trivial case of (4.12) which by no means affects the stability), although the same response function is perfectly regular if, as we did in deriving the above theorem, we *first* set $\Omega = 0$ and then let the other parameters vary.

We argue that this must be true also in more general cases. This is also another way of stating the well-known fact that a response function is a highly singular function of its space-time (or momentum–frequency) arguments and that (4.12) represents a well-defined prescription for taking the static limit, which is by no means the same as picking up a pole at some finite Ω and following it until it vanishes.

A result completely analogous to that of §§ 3 and 4 has already been derived, as mentioned in § 1, within the framework of the Landau–Fermi liquid theory (Pomeranchuk 1958, see also Nozières and Pines 1966) and is known as Pomeranchuk’s stability criterion. It has been shown (Nyberg 1968) how Pomeranchuk’s criterion can be recovered starting from the homogeneous Bethe–Salpeter equation for the vertex part. As our treatment also leads to the same equation, the results we have established become equivalent, when the appropriate limits are taken ($T \rightarrow 0$, momentum and frequency of any external probe much less than the Fermi momentum and chemical potential), to those of Pomeranchuk, of which they are the generalization to arbitrary temperatures (and momenta). They are yet more general in another sense, in so far as the kind of variation of the self-energy we consider has no restrictions coming from, say, particle–number conservation or other selection rules which instead affect the derivation of Pomeranchuk’s criterion (Nyberg 1968). This will be shown more explicitly in the next section.

5. Some simple applications

5.1. Rederivation of the Thouless–Mermin theorem

The connection between collective behaviour and thermodynamic stability has already been established limited to the Hartree–Fock approximation. It takes a more general form than our result, in so far as it states that thermodynamic stability requires the equations of the random phase approximation (i.e. the approximation for L consistent with the Hartree–Fock approximation for the grand potential) to have no complex poles; it is then not limited

to the static case. The theorem has been established by Thouless (1960) at zero temperature and generalized by Mermin (1963) to $T \neq 0$. It can be deduced from the general formulation of § 3 in the following way.

Let us restrict Δ to being of the Hartree-Fock type (cf. (3.14)) and take the basis of single-particle states which are solutions of the Hartree-Fock equation; then

$$G_{ij}(z_v) = \delta_{ij}(z_v - \mathcal{E}_i^{\text{HF}})^{-1}. \tag{5.1}$$

Moreover, as in the Hartree-Fock approximation

$$\Phi = -\frac{i}{2} \int d(1) d(2) \{G(1, 1^+)G(2, 2^+) - G(2, 1^+)G(1, 2^+)\} V(r_1 - r_2) \delta(t_1 - t_2^+). \tag{5.2}$$

We find after a few simple manipulations that Γ is frequency-independent and given by

$$\langle jn | \Gamma | im \rangle = \langle jn | V | im \rangle - \langle jn | V | mi \rangle \tag{5.3}$$

where

$$\langle ij | V | nm \rangle = \int d^3x d^3x' \varphi_i^*(x) \varphi_j^*(x') V(x - x') \varphi_n(x) \varphi_m(x'). \tag{5.4}$$

The sums over frequencies involve only G factors and can be done. The final result is

$$\delta^2 \Omega = \sum_{ij} |\Delta_{ij}|^2 \frac{f_i - f_j}{\mathcal{E}_j - \mathcal{E}_i} + \sum_{ijnm} \Delta_{ij} \frac{f_i - f_j}{\mathcal{E}_j - \mathcal{E}_i} [\langle in | V | jm \rangle - \langle in | V | mj \rangle] \frac{f_m - f_n}{\mathcal{E}_n - \mathcal{E}_m} \Delta_{mn} \tag{5.5}$$

where $f_i = f(\mathcal{E}_i)$, f being the Fermi function.

The position (leading essentially back from a self-energy to a Green function variation)

$$\Delta_{ij} = \frac{\mathcal{E}_i - \mathcal{E}_j}{f_j - f_i} \delta\rho_{ij} \tag{5.6}$$

immediately reduces (5.5) to equation (2.22) of Mermin (1963) (see also Celli and Mermin (1965) where (5.6) was first introduced). Equation (5.5), together with (4.4) written in the random phase approximation, is what is needed to prove the theorem.

The reason why, in the Hartree-Fock approximation, the conclusions are not restricted to the zero-frequency behaviour of the correlation functions can easily be traced to the vertex part being completely frequency-independent in this case; the limitations imposed on the time dependence of Δ are then unimportant, and the collective behaviour can be studied without limitations. As a consequence of (5.5) it has been proved (Mermin 1963) that a stable Hartree-Fock approximation fulfills the thermodynamic inequality (Landau and Lifshitz 1958)

$$\frac{\partial P}{\partial \rho} > 0. \tag{5.7}$$

A statement equivalent to (5.7) is

$$\left(\frac{\partial N}{\partial \mu} \right)_\beta > 0. \tag{5.8}$$

Let us use for the time being the symbols $d/d\mu$ for the total derivative with respect to μ

and $\partial/\partial\mu$ for the derivative with respect to the explicit dependence on μ . Hence, as

$$\frac{d}{d\mu} \Omega = \text{Tr} \frac{\delta\Omega}{\delta G} \frac{dG}{d\mu} + \frac{\partial\Omega}{\partial\mu} \quad (5.9)$$

(which reduces to (2.14) at equilibrium), we have

$$\left(\frac{dN}{d\mu} \right)_{\text{eq}} = -\frac{1}{\beta} \left(\frac{d^2}{d\mu^2} \Omega \right)_{\text{eq}}. \quad (5.10)$$

But

$$\frac{d^2\Omega}{d\mu^2} = \text{Tr} \left\{ \frac{\delta\Omega}{\delta G} \frac{d^2G}{d\mu^2} + \frac{dG}{d\mu} \text{Tr}' \frac{\delta^2\Omega}{\delta G \delta G'} \frac{dG'}{d\mu} + \frac{\delta}{\delta G} \frac{\partial\Omega}{\partial\mu} \frac{dG}{d\mu} \right\} + \frac{\partial^2\Omega}{\partial\mu^2}. \quad (5.11)$$

As, in any case,

$$\frac{\partial\Omega}{\partial\mu} = -\text{Tr} G \quad (5.12)$$

we then have at equilibrium

$$\text{Tr} \left\{ \frac{dG}{d\mu} \text{Tr}' \frac{\delta^2\Omega}{\delta G \delta G'} \frac{dG'}{d\mu} \right\} = -\text{Tr} \left\{ \left(\frac{\delta}{\delta G} \frac{\partial\Omega}{\partial\mu} \right) \frac{dG}{d\mu} \right\}. \quad (5.13)$$

But

$$\frac{\delta}{\delta G} \frac{\partial\Omega}{\partial\mu} = -1 \quad (5.14)$$

and hence

$$\left(\frac{dN}{d\mu} \right)_{\text{eq}} = \frac{1}{\beta} \text{Tr} \left\{ \frac{dG}{d\mu} \text{Tr}' \frac{\delta^2\Omega}{\delta G \delta G'} \frac{dG'}{d\mu} \right\} \quad (5.15)$$

which proves that a stable thermal equilibrium solution cannot violate the thermodynamic inequality (5.8) (compare (5.15) with (3.1)). This result too is found to be true for any conserving approximation and not limited to the Hartree-Fock approximation. If we take the $T \rightarrow 0$ limit of (5.15), (5.8) becomes identical with the first of Pomeranchuk's conditions, i.e. $1 + F_0 > 0$ (see Nozières and Pines (1966) for the meaning of the symbols).

5.2. Cooper instability

In order to treat the superconducting instability we must generalize slightly the formalism used up to now. Let us recall that superconductivity is best described by using Nambu's formalism (Nambu 1960, Schrieffer 1964), in which the self-energy is a 2×2 matrix

$$\Sigma(\mathbf{p}, z_\nu) = z_\nu \{1 - Z(\mathbf{p}, z_\nu)\} + \chi(\mathbf{p}, z_\nu) \tau_3 + \phi(\mathbf{p}, z_\nu) \tau_1 \quad (5.16)$$

(τ_1, τ_3 are the usual Pauli matrices) and G and Σ are related by Dyson's equation (a matrix equation now) with

$$G_0^{-1}(\mathbf{p}, z_\nu) = z_\nu - \mathcal{E}_p \tau_3. \quad (5.17)$$

ϕ is the order parameter, to be determined self-consistently, and $\phi \equiv 0$ corresponds to the normal state. In this case G is given by

$$G(\mathbf{p}, z_\nu) = \frac{z_\nu Z(\mathbf{p}, z_\nu) + \mathcal{E}_p \tau_3}{\{z_\nu Z(\mathbf{p}, z_\nu)\}^2 - \mathcal{E}_p^2}. \quad (5.18)$$

Going back to § 2, let us assume that the indices i, j, \dots correspond now to the matrix indices

of Nambu, and take for a translationally invariant system

$$\Delta_{ij}(z_\nu) = \phi(\mathbf{p}, z_\nu)(\tau_1)_{ij}. \tag{5.19}$$

Products such as $G\Delta G$ (cf. (3.5)), where G is chosen to represent the normal-state solution (5.18), become then

$$(G\Delta G)_{\alpha\delta} = \phi(\mathbf{p}, z_\nu)D(\mathbf{p}, z_\nu)(\tau_1)_{\alpha\delta} \tag{5.20}$$

$$D(\mathbf{p}, z_\nu) = [\{z_\nu Z(\mathbf{p}, z_\nu)\}^2 - \mathcal{E}_p^2]^{-1} \tag{5.21}$$

and the expression for $\delta^2\Omega$ can be cast into the form

$$\begin{aligned} \delta^2\Omega = & \frac{1}{\beta} \sum_{\nu, \mathbf{p}} \left\{ \frac{1}{\beta} \sum_{\nu', \mathbf{p}'} \phi(\mathbf{p}, z_\nu)D(\mathbf{p}, z_\nu) \frac{1}{2} \sum_{impq} \left[(\tau_1)_{im} \left(\frac{\delta^2\phi}{\delta G_{im} \delta G_{qp}} \right)_{U=0} (\mathbf{p}, \mathbf{p}'; z, z_{\nu'}) (\tau_1)_{qp} \right] \right. \\ & \left. \times \phi(\mathbf{p}', z_{\nu'})D(\mathbf{p}', z_{\nu'}) - \phi^2(\mathbf{p}, z_\nu)D(\mathbf{p}, z_\nu) \right\}. \end{aligned} \tag{5.22}$$

The second variational derivative of ϕ appearing in (5.22) must be understood as being taken in the presence of the appropriate external source field (which will couple in this case with $\psi^+(1)\psi^+(2)$ and $\psi(1)\psi(2)$ and is then the generalization appropriate to the present case of U in §2). Then, if we set $U = 0$, the result must be Fourier transformed with respect to the time variables and taken at $\Omega_m = 0$ (and also, as we have specialized to a translationally invariant case and Δ is diagonal as far as momentum is concerned, the result will depend only on two independent momenta). Before setting $U = 0$, the expression within square brackets in (5.22) is (we do not indicate explicitly the space-time variables in order not to lengthen the formulae)

$$\frac{1}{2} \sum_{impq} [\dots] = \frac{1}{2} \left\{ \frac{\delta^2\phi}{\delta G_{12} \delta G_{21}} + \frac{\delta^2\phi}{\delta G_{21} \delta G_{12}} + \frac{\delta^2\phi}{\delta G_{12} \delta G_{12}} + \frac{\delta^2\phi}{\delta G_{21} \delta G_{21}} \right\} \tag{5.23}$$

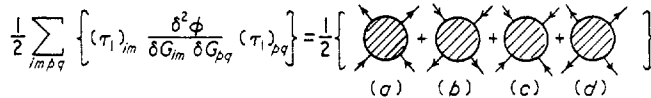


Figure 4. Graphical illustration of (5.23).

and contains diagrams of the four types displayed in figure 4. It is clear that, when $U \rightarrow 0$, only diagrams (a) and (b) (i.e. diagrams with equal numbers of particles entering and leaving) are left; as (a) and (b) only differ by an interchange of (p, ν) with (p', ν') , which is unimportant owing to the symmetry of (5.22), it is clear that the final outcome of this (only formally complex) operation is just the particle-particle irreducible vertex for two particles with zero total energy and momentum. This is what we needed, as it is just an instability in the particle-particle channel of the vertex part which leads to the superconducting transition (Schrieffer 1964).

Then let $\Gamma(\mathbf{p}\mathbf{p}'; z_\nu z_{\nu'})$ be the vertex; (5.22) becomes

$$\delta^2\Omega = \frac{1}{\beta^2} \sum_{\nu\nu'} \sum_{\mathbf{p}\mathbf{p}'} \phi^*(\mathbf{p}, z_\nu) \hat{X}(\mathbf{p}\mathbf{p}'; z_\nu z_{\nu'}) \phi(\mathbf{p}', z_{\nu'}) \tag{5.24}$$

with

$$\hat{X}(\mathbf{p}\mathbf{p}'; z_\nu z_{\nu'}) = D(\mathbf{p}, -\nu)\Gamma(\mathbf{p}\mathbf{p}'; -\nu, \nu')D(\mathbf{p}', \nu') - \delta_{\mathbf{p}\mathbf{p}'}\beta\delta_{-\nu, \nu'}D(\mathbf{p}, -\nu). \tag{5.25}$$

The proof given of the basic properties (3.21) can be extended without difficulty to the present case; hence the discussion of the preceding section can be taken up (with only minor changes). In particular (4.13) now becomes equivalent to the homogeneous Bethe-Salpeter equation for the particle-particle vertex at zero total energy-momentum of the particle pair.

The temperature at which it first has a non-trivial solution is the superconducting transition temperature, and is identical with the temperature at which $\delta^2\Omega$ first becomes zero (i.e. the temperature at which the normal state and the superconducting state with an infinitesimal amplitude of the gap parameter have the same free energy). Had we started from below T_c (4.13) would give the limiting value of the gap equation as $T \rightarrow T_c$.

We have studied in detail the superconducting instability only in order to show how general the method developed in §§ 3 and 4 can be. Other types of instabilities could be devised (e.g. the Overhauser spin density wave instability (Celli and Mermin 1965, Celli *et al.* 1966, Celli and Morandi 1967), but the study proceeds essentially along the same lines and would not then be especially illuminating. We refer for them to the literature (see, mainly, Mattuck and Johansson 1968).

6. Concluding remarks

The two approaches to the instability problem, the dynamical approach looking at the appearance of zero-frequency poles in the appropriate correlation functions (or, equivalently, in the corresponding vertex parts) and the statistical-mechanical approach, where instabilities appear as a failure of a given approximation to the free energy to be a local minimum, have been found to be completely equivalent; this gives a general justification to the way many-body theorists look at the onset of instabilities. Apart from the possibility of trivial examples, the solutions of (4.13) (which are, as we saw, connected with the onset of such instabilities) are always, loosely speaking, off-diagonal in some quantum numbers with respect to which the normal solution is diagonal (particle number in the case of Cooper's instability, spin and momentum in the case of Overhauser's spin density waves, etc.). We (eventually) found that the temperature at which these non-trivial solutions begin to exist is the transition temperature towards a new phase, characterized by some order parameter which is intimately connected with the solution of (4.13) (again, referring to the above examples, the order parameter is $\langle \psi_{K\uparrow} + \psi_{-K\downarrow} \rangle$ for a superconductor, $\langle \psi_{K+Q,\uparrow} + \psi_{K\downarrow} \rangle$ in the spin density wave case, and so on). The structure of the solution then immediately gives a hint on how to treat the system below the transition temperature. What we have to do is just to introduce additional Green functions corresponding to the new non-vanishing expectation values (the 'anomalous' Green function of Gor'kov and Nambu in the theory of superconductivity) and to re-adapt the theoretical scheme of § 2 to this new situation. The formal extension of the equations of § 2 to the more general case in which the Green function is a matrix (with off-diagonal elements corresponding to the 'anomalous' functions) is easy and without special problems. This technique is well known in the case of superconductivity; it has been also applied to the spin density wave problem (Celli and Morandi 1967). A general review of it has been given by Mattuck and Johansson (1968) from the diagrammatic point of view.

Having in mind some definite phase transition, one could start from the very beginning with the above formalism, obtain self-consistent equations for the anomalous Green function and determine the transition temperature (or equivalently, at fixed temperature, the critical interaction strength) as the temperature at which their solution vanishes. This is the point of view of Mattuck and Johansson (1968). It has been shown by the same authors that, at the critical point, the equation for the anomalous part of the self-energy reduces to the homogeneous Bethe-Salpeter equation for the vertex part taken in the appropriate channel, which is also our result. The two points of view, the latter and that of the present paper, then just complement each other: our approach (which considers the problem, so to speak, from the normal side, or from above the critical point) goes smoothly, as the critical point is crossed, into the renormalized theory developed by Mattuck and Johansson; the two approaches together give, we believe, a fairly complete analysis of the possible second-order phase transitions in a Fermi system.

Appendix. Properties of the irreducible vertex part at zero total energy

Using (3.19) it can immediately be checked that (3.21a) and (3.21b) are true provided

$$\langle ik | \Gamma_{\mu\mu'} | jl \rangle = \langle lj | \Gamma_{-\mu', -\mu} | ki \rangle^* \quad (\text{A1})$$

and

$$\langle ik | \Gamma_{\mu\mu'} | jl \rangle = \langle jl | \Gamma_{-\mu, -\mu'} | ik \rangle^* \tag{A2}$$

Using (A1), (A2) can be reduced to

$$\langle ik | \Gamma_{\mu\mu'} | jl \rangle = \langle ki | \Gamma_{\mu'\mu} | lj \rangle \tag{A3}$$

(A1) and (A3) are graphically represented in figure 5.

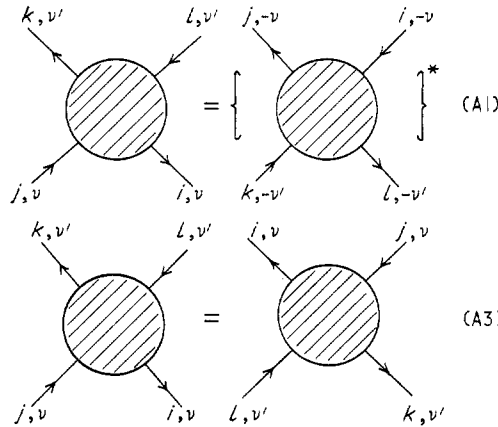


Figure 5. Graphical representation of equations (A1) and (A3).

That the two diagrams of (A3) are equal is immediately evident: the right-hand side of the graphical equation just corresponds to a rotation of the left-hand side around its centre; the two are topologically the same diagram. As to (A1), let us suppose we analyse the diagrams into skeletons; the only change to the standard rules of finite temperature perturbation theory is that to each line in a diagram we must associate a dressed propagator $G_{ij}(z_\nu)$ (and to G_{ij} we associate a line running from j to i) instead of a bare one (Abrikosov *et al.* 1963). From these rules and the hermiticity condition on G , we see that complex conjugation just changes the sign of the four external frequency variables (the internal ones are also changed, but they are dummy indices) and inverts the direction of each propagator line. After these operations have been performed, it is then also proved that every diagram on the right-hand side of (A1) turns into a diagram contributing to the left-hand side of (A1).

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